Math 242: Project 2 The Logistic Model

If ecologists were ever asked to write a new Decalogue, their First Commandment would be: Thou shalt not transgress the carrying capacity. Garrett Hardin

In Chapter 3 we studied several models of exponential growth and decay. Now we will see how to adapt these models to describe quantities that approach some limiting value over time due to the influence of external factors, such as the height of a sunflower or when a population begins to exhaust available resources.

CASE STUDY: Sunflower Growth

Table 1 shows the height of a sunflower after a certain number of growing days. If we let x represent the number of growing days and let $H(x)$ represent the height of the sunflower, then a good model for this data is given by the function

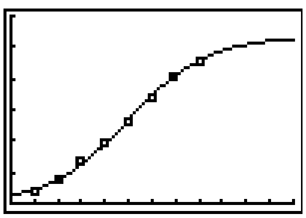
$$H(x) = \frac{266.51}{1+19.70e^{-0.0856x}}$$

This function is an example of a *logistic model*. The value in the numerator represents the limiting height of the sunflower, or 266.51 centimeters.

Table 1

Growing Days	Height (cm)
7	17.93
14	36.36
21	67.76
28	98.10
35	131.00
42	169.50
48	205.50
56	228.30

Source: Proceedings of the National Academy of Sciences, 1919



[0, 84] by [0, 300]

Figure 1

A graph of the data and the model is shown in Figure 1. Notice how the graph initially exhibits exponential growth. At first, the sunflower grows more and more rapidly. However, at some point the growth begins to slow down. As the head of the sunflower matures, it becomes very heavy and the narrow stalk reaches its maximum capacity to support such a large weight.

THE LOGISTIC MODEL

Most populations cannot grow exponentially for an unlimited period of time. At some point external factors or limited resources will restrict further growth. For example, as a population becomes large it may begin to exhaust the available food supply or natural resources. As a result, the growth rate will begin to slow, and eventually the population may approach a standstill.

The Belgian mathematician Pierre François Verhulst proposed a modified form of the Malthusian population model that incorporates the concept of self-limiting growth due to competition for resources. His model assumes that there is a maximum sustainable population, called the **carrying capacity**, that sets a limit as to how large a population can grow. If we let L be the carrying capacity and P the population at time t, then one way to derive the Verhulst model is to consider the ratio

Pierre Verhulst

$$\frac{L-P}{P}$$

which is essentially a measure of the relative capacity for growth. If P is small in comparison to L, this ratio will be large indicating that there are plenty of available resources to sustain exponential growth at first. As time passes and the population increases, competition for resources becomes a significant factor and the capacity for growth is diminished. Ultimately $P \to L$ and this ratio shrinks zero asymptotically.

Note: In calculus, the Verhulst model can be expressed as the *differential equation*

$$\frac{dP}{dt} = kP(L - P)$$

which, when solved, yields the logistic function. As we saw in Section 2.1, the rate of growth (dP/dt) is a quadratic function of the population P. Now we are considering the population as a function of time.

Thus, it seems reasonable to assume that the ratio $(L - P)/P$ can be modeled by an exponential decay function. That is,

$$\frac{L-P}{P} = ce^{-kt}$$

where c and k are positive constants. Solving this equation for P leads to the following function, which is known as the *logistic model*.

LOGISTIC MODEL

The **logistic model** is given by

$$P(t) = \frac{L}{1 + ce^{-kt}}$$

where P is the population at time t, $L > 0$ is the carrying capacity, $k > 0$ is the continuous decay rate, and $c = (L - P_0)/P_0$.

EXAMPLE 1 *Application: Population of Indonesia*

The following table shows the total population of Indonesia for various years between 1950 and 2004. A logistic model for this data is given by

$$P(t) = \frac{567.8}{1 + 21.4e^{-0.0256t}}$$

where P is the population , in millions, t years from 1900 (so that $t = 0$ corresponds to 1900). *(Source: U.S. Census Bureau)*

Year	Population (millions)
1950	83.0
1960	100.1
1970	122.3
1980	150.5
1990	181.8
2000	213.8
2004	226.0

(a) According to the model, what was the population of Indonesia in 1900 and in 2000? What is the percent error for the 2000 figure when compared with the actual value?

(b) Determine the end behavior of the model algebraically. What is the carrying capacity for the population of Indonesia?

(c) Use technology to graph the model and the data together.

(d) According to the model, when will the population of Indonesia reach 95% of its carrying capacity?

SOLUTION

(a) For 1900 we evaluate the function for $t = 0$, and for 2000 we use $t = 100$.

$$P(0) = \frac{567.8}{1 + 21.4e^{-0.0256(0)}} = \frac{567.8}{1 + 21.4} \approx 25.3 \text{ million}$$

$$P(100) = \frac{567.8}{1 + 21.4e^{-0.0256(100)}} \approx \frac{567.8}{1 + 21.4(0.0773)} \approx 213.2 \text{ million}$$

To find the percent error in the 2000 figure, we divide the difference in the population figures by the actual population:

$$\frac{213.2-213.8}{213.8}=-0.0028$$

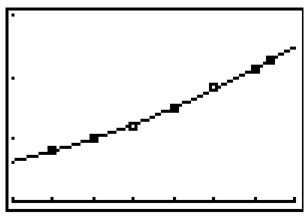
Thus, the value from the model is off by only 0.28%.

(b) As $t \to -\infty$, $e^{-0.0256t} \to \infty$. Thus, for large negative t, the denominator of the function grows without bound, so that the fraction itself will go to zero. The function has a horizontal asymptote at $P = 0$. As $t \to \infty$, $e^{-0.0256t} \to 0$. Thus, for large positive t, the denominator of the function will approach a value of $1 + 21.4(0) = 1$, so that the fraction itself will approach 567.8. The function has a second horizontal asymptote at $P = 567.8$. Summarizing the two results, we have

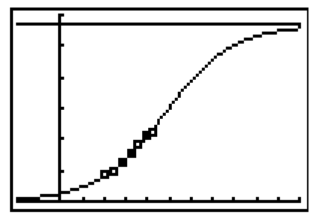
Left end-behavior: As $t \to -\infty$, $P \to 0$
Right end-behavior: As $t \to \infty$, $P \to 567.8$

According to the model, the limiting value on the population is 567.8 million. This figure is the carrying capacity. Notice that it appears in the numerator of the function.

(c) In Figure 2 we have used a graphing calculator to graph the model with the data. You can see how well the logistic curve seems to fit the data points. In Figure 3 we have expanded the viewing rectangle to accommodate a longer period of time so that the end behavior can be clearly seen. The horizontal line is the asymptote $y = 567.8$.



[40, 110] by [0, 300]
Figure 2



[−50, 275] by [0, 600]
Figure 3

(d) We let $P = 0.95(567.8) = 539.4$ and solve for t.

$$539.4=\frac{567.8}{1+21.4e^{-0.0256t}} \quad \text{Substitute } P = 539.4$$

$$539.4(1+21.4e^{-0.0256t})=567.8 \quad \text{Multiply by the denominator}$$

$$539.4+11543.16e^{-0.0256t}=567.8 \quad \text{Distribute}$$

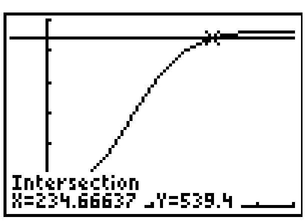
$$11543.16e^{-0.0256t}=28.4 \quad \text{Subtract 539.4}$$

$$e^{-0.0256t}=0.00246 \quad \text{Divide by 11543.16}$$

$$-0.0256t=\ln 0.00246 \quad \text{Take ln of both sides}$$

$$t=-\frac{\ln 0.00246}{0.0256} \quad \text{Isolate } t$$

$$\approx 234.7$$



[−50, 350] by [0, 600]
Figure 4

Since 1900 + 234.7 = 2134.7, the population of Indonesia is predicted to reach 95% capacity in the year 2134.

To solve this problem graphically, we graphed the constant function $P = 539.4$ and determined the first coordinate of the point of intersection, as shown in Figure 4. ■