Isometries of the Plane and Symmetry Groups of Regular Polygons

Definition: An isometry is a transformation of the plane which preserves distances between points.

\[ \text{Iso}(\mathbb{R}^2) = \{ S: \mathbb{R}^2 \to \mathbb{R}^2 \mid d(S(p), S(q)) = d(p, q) \} \] where \( d \) is the Euclidean distance

Because isometries preserve distances the spatial relations between points and figures stays fixed which results in the following:

Claim 1: There are 4 Types of Isometries:

1) Rotations, \( R_\theta (p) \) (center \( p \), angle \( \theta \)): any direct isometry with a fixed point, \( p \)
2) Translations, \( T_v \) (along vector \( v \)): any direct isometry with no fixed point
3) Reflections, \( R_l \) (through line \( l \)): any opposite isometry with a fixed point
4) Glide reflections, \( G_{l,v} = T_v \circ R_l \): any opposite isometry with no fixed point

direct: “orientation-preserving”
opposite: “orientation-reversing”

Note: Every isometry can be written as (a composition of) 1, 2, or 3 reflections.

a. Rotations: \( R_\theta (p) = R_{l'} \circ R_{l'} \) thru any two lines that intersect at \( p \) at an angle half that of the rotation

b. Translations: \( T_v = R_{l'} \circ R_{l'} \) thru any two lines both perpendicular to \( v \) and separated by half its magnitude

Note: let \( p \to \infty \) in (a), then a translation is a rotation with center \( p = \infty \)

c. Reflections: \( R_l \)

d. Glide reflections: \( G_{l,v} = T_v \circ R_l = (R_{l'} \circ R_{l''}) \circ R_l \)
The **Symmetry Group of X**: \( \text{Symm}(X) = \{ S \in \text{Iso}(\mathbb{R}^2) \mid S(X) = X \} \) where \( X \) is any subset of \( \mathbb{R}^2 \).

**Theorem 1**: For regular polygons \( P_n \) with \( n \geq 3 \),

\[
\text{Symm}(P_n) = \{ I = R_0, R_{2\pi/n}, \ldots, R_{2(n-1)\pi/n}, R_{L_1}, \ldots, R_{L_n} \}
\]

Rotations thru vertices Reflections thru lines joining about the centroid (origin) vertices and/or midpoints

Justification: Since any isometry in \( \text{Symm}(P_n) \) must leave \( P_n \) invariant and also preserve distances between points, it must map vertices to vertices, midpoints to midpoints, etc. I.e. it must be a permutation of the set of vertices and the set of midpoints. This excludes translations and glides since at least one of the points would get mapped outside of the set in those cases. That leaves two types of isometries: rotations about the centroid of the polygon and reflections through lines as follows

![Diagram of rotations and reflections for regular polygons](image)

**Algebraically**: let \( a = R_{2\pi/n} \), \( b = R_{L_1} \)

\[
(\text{then } a^k = R_{2k\pi/n}, \quad a^n = a^0 = b^2 = I, \quad \text{and } a^k b = R_{L_i} \text{ for some } i = 0, 1, \ldots, n)
\]

**cyclic group of order \( n \)**: \( C_n = \langle a \rangle = \{ I, a, \ldots, a^{n-1} \} \)

**dihedral group of order \( n \)**: \( D_n = \langle a, b \rangle = \{ I, a, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b \} \) possibly permuted

\( D_n = \text{Symm}(P_n) = \{ R_0, R_{2\pi/n}, \ldots, R_{2(n-1)\pi/n}, R_{L_1}, \ldots, R_{L_n} \} \)

**Theorem 2**: Every finite subgroup \( \text{Iso}(\mathbb{R}^2) \) of is isomorphic to \( C_n \) or \( D_n \)!

If \( G \subset \text{Iso}(\mathbb{R}^2) \) is a finite subgroup then \( G \) fixes a point \( p_0 \in \mathbb{R}^2 \) and is either

1. \( C_n = \langle R_{2\pi/n}(p_0) \rangle \) or
2. \( D_n = \langle R_{2\pi/n}(p_0), R_l \rangle \) for some line \( l \) through \( p_0 \).
Proof of Claim 1:

Case 1:  $S$ is a direct isometry with a fixed point, $p_0 \Rightarrow S$ is a rotation.

Consider any other point $q$ and its image:

$q$ and $S(q)$ are equidistant from $p$:
\[
d(q, p_0) = d(S(q), S(p_0)) = d(S(q), p_0)
\]

Thus $S$ must be $R_\theta(p_0)$ since

\[
S \rightarrow R_\theta(p_0)^{-1}
\]

$R_\theta(p_0)^{-1} \circ S$ fixes $q$ and $p_0$ and must be either a reflection or the identity (since translations and rotations cannot have two fixed points.) But a composition of two direct isometries cannot be a reflection, which is indirect. Thus $R_\theta(p_0)^{-1} \circ S = I$ so $S = R_\theta(p_0)$ is a rotation.

Case 2:  $S$ is a direct isometry with no fixed points \( \Rightarrow S \) is a translation.

Consider $v$ from $p$ to $S(p)$:

\[
S \rightarrow T_v \rightarrow S(p)
\]

$S = T_v \circ R_\theta(p_0)$ would have a fixed point at $q$ in the diagram, contradicting our assumption, unless $R_\theta(p_0)=I$ which means $S = T_v$ is a translation.

Case 3:  $S$ is an opposite isometry with a fixed point, $p_0 \Rightarrow S$ is a reflection.

Consider any line $L$ through $p_0$. Then $R_L \circ S$, a composition of two opposite isometries, is a direct isometry with a fixed point and so must be a rotation, $R_\theta(p_0)$, by case 1. Then there exists a line $L'$ through $p_0$ that forms an angle of $\theta/2$ with line $L$ so that $R_\theta(p_0) = R_L \circ R_{L'}$ as noted on page 1. $S$ must be $R_{L'}$ since $R_L \circ S = R_\theta(p_0)$ means $S = R_{L'}^{-1} \circ R_\theta(p_0) = R_L \circ R_\theta(p_0)$ (since $R_L = R_{L'}^{-1}$) and so we end up with $S = R_L \circ R_L \circ R_{L'} = I \circ R_{L'}$ so $S = R_{L'}$ is a reflection.
Case 4: S is an opposite isometry with no fixed point \( \Rightarrow \) S is a glide reflection.

Consider the segment connecting any \( p \) with \( S(p) \), and let \( q \) be the midpoint:

\[
\begin{array}{c}
\text{p} \\
\text{q} \\
\text{S(p)}
\end{array}
\]

a) If \( S(q) \) is collinear with \( p \) and \( q \) (on line \( L \) say) then \( S \) leaves \( L \) invariant because it preserves distances: suppose \( r \) is another point on \( L \), then the distances between \( p, q, r \) will be the same as the distances between \( S(p), S(q), S(r) \) respectively (i.e. \( \Delta pqr \cong \Delta S(p)S(q)S(r) \) by SSS Congruence), so if \( r \) is collinear with \( p \) and \( q \) (a flat triangle) then \( S(r) \) will be collinear with \( S(p) \) and \( S(q) \) which under this assumption are all on line \( L \) with \( p \) and \( q \).

So then \( RL \circ S \) leaves \( L \) invariant since each one does, and is direct since it’s a composition of two opposite isometries. But by case 2 that means it is a translation, \( RL \circ S = T_v \), and so \( S = RL \circ T_v = GL_v \) is a glide reflection. (we used \( RL = RL^{-1} \)).

b) If \( S(q) \) is not collinear with \( p \) and \( q \). Then consider line \( L \) connecting \( q \) and \( S(q) \) and the projections of \( p \) and \( S(p) \) onto \( L \) (the dotted segments ending at right angles to \( L \) at points \( r \) and \( t \).) We get the following congruent triangles:

\[
\begin{array}{c}
\Delta pqr \cong \Delta S(p)qt \text{ by ASA (vertical angles at } q, \text{ right angles, and } q \text{ midpoint)}
\end{array}
\]

\[
\begin{array}{c}
\Delta S(p)qt \cong \Delta S(p)S(q)t \text{ by SSS or SAS (right angles, } d(S(p),q) = d(p,q) = d(S(p),S(q)) \text{ because } q \text{ midpoint and } S \text{ isometry, shared side, then Pythagorean Thm to get third sides congruent)}
\end{array}
\]

So then \( \Delta pqr \cong \Delta S(p)S(q)t \) transitively (and the triangles are oppositely oriented.) But we also know that \( \Delta pqr \cong \Delta S(p)S(q)S(r) \) since \( S \) is an isometry. So it must be that \( S(r) = t \) because the only other option for \( S(r) \) would leave \( \Delta S(p)S(q)S(r) \) oriented the same as \( \Delta pqr \) but it needs to be oppositely oriented, since \( S \) is an opposite isometry, which has to be \( \Delta S(p)S(q)t \).

Then let \( v \) be the vector from \( q \) to \( S(q) \). \( \textbf{S must be } GL_v \textbf{ since } GL_v^{-1} \circ S \textbf{ fixes } \Delta pqr \textbf{ and any isometry that fixes three distinct points must be the identity, so } GL_v^{-1} \circ S = 1 \Rightarrow S = GL_v \textbf{ is a glide reflection.} \)
Proof of Theorem 2: Every finite subgroup $G$ of $\text{Iso}(\mathbb{R}^2)$ fixes a point and is isomorphic to $C_n$ or $D_n$.

Let $G = \{S_1, S_2, \ldots, S_n\}$ and $p \in \mathbb{R}^2$.

The “orbit of $G$ through $p$” is $G(p) = \{S_1(p), S_2(p), \ldots, S_n(p)\} = \{p_1, p_2, \ldots, p_n\}$

Any $S$ in $G$ will permute the set $\{p_1, p_2, \ldots, p_n\}$ because $S(p_i) = (S \circ S_i)(p) = S_j(p)$ for some $j$ since $G$ is a group and $j \neq i$ unless $S$ is the identity since an isometry must keep the distances invariant so it must keep the points in order. This rules out translations and glides since they cannot permute a finite set.

The centroid of the orbit, $p_0 = \frac{p_1 + \cdots + p_n}{n}$, must be a fixed point for any $S$ in $G$.

I don’t know exactly how to show that $S(p_0) = S\left(\frac{p_1 + \cdots + p_n}{n}\right) = \frac{p_1 + \cdots + p_n}{n} = p_0$ since rotations are not linear unless the center is at the origin. However, $S(p_0)$ must be the same distance from each of the $p_i$ as $p_0$ since $S$ is an isometry and it permutes the $p_i$.

a. If $S$ is direct it must be a rotation about $p_0$ (direct with fixed point $p_0$)

Since $G$ is finite let the smallest rotation in $G$ be $R_\theta$. Then for $R_\alpha$ in $G$, by the division algorithm we have $\alpha = m\theta + \beta$ where $\beta < \theta \Rightarrow \beta = 0$ so $R_\alpha = R_{m\theta} = (R_\theta)^m$. Therefore $R_\theta$ generates all of the rotations in $G$, and if $G$ contains rotations only then $G = \langle R_\theta \rangle = C_n$.

b. If $S$ is opposite then it must be a reflection through a line containing $p_0$, say $L$. (opposite with fixed point $p_0$)

Let $G' = \langle R_0, R_L \rangle$ where $R_0$ is the smallest rotation in $G$. As above all of the rotations are contained in $G'$, we just need that if $R_{L'} \in G$ then $R_{L'} \in G'$ to get that $G'$ is all of $G$. Well $R_L \circ R_{L'} = R_\alpha \in G$ since $G$ is a group and a composition of two reflections through nonparallel lines is a rotation. Then $R_L \circ R_{L'} = (R_0)^k$ for some $k < n \Rightarrow R_{L'} = (R_0)^k \circ R_L \in G$.

So if $G$ contains both reflections and rotations $G = \langle R_\theta, R_L \rangle = D_n$. This would include the case $G = \langle R_L \rangle = \langle R_0, R_L \rangle = \{I, R_L\} = D_1 = C_2$. 